

Aniq integral

12.7.1. Aniq integral tushunchasiga olib keladigan masalalar

1. Egri chiziqli trapetsiya yuzini hisoblash masalasi. Avvalo, egri chiziqli trapetsiya tushunchasini kiritamiz. Aytaylik, $y=f(x)$ funksiya $[a,b]$ kesmada aniqlangan uzluksiz va manfiy bo'lmasin. U holda $x=a$, $x=b$, $y=0$ va $y=f(x)$ chiziklar (grafiklar) bilan chegaralangan shaklni *egri chiziqli trapetsiya* deb ataymiz (12.7.1 – rasmga qarang).

Shu egri chiziqli trapetsiya yuzini hisoblash masalasini qo'yaylik.

Ma'lumki, agar $y=f(x)$ chiziqli funksiya bo'lsa, bu egri chiziqli trapetsiya oddiy to'g'ri burchakli trapetsiyaga aylanib qoladi va bu hol uchun masala haldir.

Umumiy holda talab qilingan yuzani hech bo'lmaganda taqribiy hisoblash masalasini qaraymiz. Buning uchun $[a,b]$ kesmani ixtiyoriy n ta bo'laklarga ajratamiz va bo'linish nuqtalarini o'sib borish tartibida quyidagicha yozamiz:

$$a=x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

Endi, har bir bo'linish nuqtalarining $y_i=f(x_i)$ ($i=0,1,\dots,n$) ordinatalari yordamida egri chiziqli trapetsiyani n ta bo'laklarga ajratamiz (12.7.2 – rasmga qarang).

Egri chiziqli trapetsiyaning $[a;b]$ kesmaning uzunligi $\Delta x_i = x_i - x_{i-1}$ bo'lgan $[x_{i-1}, x_i]$ qismiga mos keluvchi i – bo'lagining yuzini ΔS_i bilan belgilab, uning taqribiy qiymatini hisoblash maqsadida $\xi_i \in [x_{i-1}, x_i]$ nuqtani olamiz va asosi Δx_i dan, balandligi esa $f(\xi_i)$ ordinatadan iborat to'g'ri to'rtburchakni quramiz va

$$\Delta S_i \approx f(\xi_i) \cdot \Delta x_i$$

deb qabul qilamiz. Bu egri chiziqli trapetsiya i – bo'lagi yuzining taqribiy qiymatidir. Bu ishni barcha bo'lakchalar uchun bajarib, egri chiziqli trapetsiyaning yuzi S uchun

$$S = \sum_{i=1}^n \Delta S_i \approx \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (12.7.1)$$

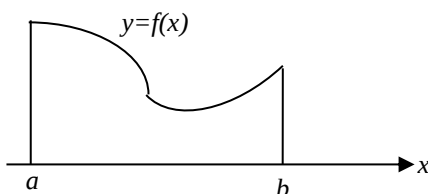
taqribiy formulaga ega bo'lamiz.

Endi, $\lambda = \max_i \Delta x_i$ ($i=1;2,\dots,n$) deb belgilaylik. Agar $\lambda \rightarrow 0$ ($\Rightarrow n \rightarrow \infty$) (12.7.1)

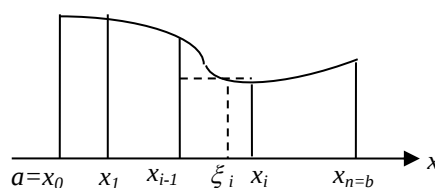
ning o'ng tomonidagi yig'indining chekli limiti mavjud bo'lib, bu limit $[a,b]$ kesmani n ta bo'laklarga bo'lish usuliga va har bir i - bo'lakdan olinayotgan ξ_i ning o'rniga bog'liq bo'lmasa, uni egri chiziqli trapetsiyaning yuzi deb qabul qilamiz. Shunday qilib, egri chiziqli trapetsiyaning yuzi uchun

$$S = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (12.7.2)$$

formulani olamiz.



12.7.1-pacm.



12.7.2-pacm.

2.To'g'ri chiziqli yo'l bo'ylab o'zgaruvchi kuchning bajargan ishini hisoblash masalasi. Aytaylik, moddiy nuqta berilgan to'g'ri chiziqli yo'l bo'ylab $q(x)$ o'zgaruvchi kuch ta'sirida harakatlanayotgan bo'lsin, hamda $q(x)$ funksiya $[a,b]$ kesmada uzluksiz va manfiy emas deb faraz qilaylik.

Ma'lumki, agar $[a,b]$ kesma bo'ylab o'zgarmas q kuch ta'sirida moddiy nuqta harakatlanayotgan bo'lsa, uning bajargan ishi $A=q(b-a)$ dan iboratdir. Agar kuch o'zgaruvchi bo'lsa, bu formula yaroqsizdir. Bu holda, bajarilgan ishning taqribiy qiymatini hisoblash maqsadida oldingi badda qilingan ishlarni takrorlaymiz va i-bo'lakda bajarilgan ishning taqribiy qiymati uchun

$$\Delta A_i \approx q(\xi_i) \cdot \Delta x_i$$

ni olamiz va $[a,b]$ kesmada bajarilgan ishning taqribiy qiymati uchun

$$A = \sum_{i=1}^n \Delta A_i \approx \sum_{i=1}^n q(\xi_i) \Delta x_i \quad (12.7.3)$$

ga ega bo'lamiz. Bu yerda ham $\lambda \rightarrow 0$ dagi limitga o'tish bilan

$$A = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n q(\xi_i) \Delta x_i \quad (12.7.4)$$

formulaga kelimiz.

3.Kesma bo'ylab tarqalgan massani hisoblash masalasi. Bu yerda $[a,b]$ kesma bo'ylab o'zgaruvchi $\rho(x)$ zichlik bilan tarqalgan massani hisoblash masalasini qo'yamiz. Yuqorida qilingan ishlarni takrorlab, $[a,b]$ kesmada tarqalgan m massaning taqribiy qiymati uchun

$$m \approx \sum_{i=1}^n \rho(\xi_i) \Delta x_i \quad (12.7.5)$$

formulani, aniq qiymati uchun esa

$$m = \lim_{\lambda \rightarrow 0} \sum_{i=1}^m \rho(\xi_i) \Delta x_i \quad (12.7.6)$$

ni olamiz.

12.7.2. Aniq integralning ta'rifi

Yuqorida ko'rilgan uchchala masalada ham (12.7.1), (12.7.3) va (12.7.5) taqribiy formulalarning o'ng tomonlari bir xildir. Aniq formulalar bo'lgan (12.7.2), (12.7.4) va (12.7.6) larda ham xuddi shundaydir.

Endi, bu yerda $[a,b]$ kesmada aniqlangan va chegaralangan $f(x)$ funksiyani qaraymiz. Yuqorida qilingan ishlarni takrorlab,

$$\sigma_n = \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (12.7.7)$$

yig'indini tuzamiz. Bu $f(x)$ funksiyaning $[a,b]$ oraliq bo'yicha integral yig'indisi deb ataladi, bu yerda n $[a,b]$ kesmani bo'lishlar soni, $\Delta x_i = x_i - x_{i-1}$, $a = x_0 < x_1 < \dots < x_n = b$ kesmaning bo'linish (tugun) nuqtalari, $\xi_i \in [x_{i-1}, x_i]$ ixtiyoriy nuqta.

Yuqoridagidek, $\lambda = \max_i \Delta x_i$ ($i = 1; 2; \dots; n$) deb belgilaymiz.

Agar $f(x)$ funksiyaning $[a, b]$ oraliq bo'yicha integral yig'indisining $\lambda \rightarrow 0$ dagi chekli limiti mavjud bo'lib, u oraliqni n ta bo'laklarga bo'lish usuliga hamda i – oraliqdan olingan ξ_i nuqtaning o'rniga bog'liq bo'lmasa, bu limit $f(x)$ funksiyaning $[a, b]$ oraliq bo'yicha aniq integrali deyiladi va $\int_a^b f(x) dx$ kabi yoziladi. Bu holda $f(x)$ funksiya $[a, b]$ oraliqda integrallanuvchi deyiladi.

Bu yozuvda a – aniq integralning quyi, b esa yuqori chegarasi, $f(x)$ integral osti funksiyasi, $f(x)dx$ integral osti ifodasi, x integral o'zgaruvchisi, $\int$ - integral belgisidir. $[a, b]$ ni integrallash oralig'i deyiladi.

Demak, ta'rif bo'yicha (12.7.7) da $\lambda \rightarrow 0$ dagi limitga o'tib,

$$\int_a^b f(x) dx = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \quad (12.7.8)$$

ga egamiz.

Yuqorida tuzilgan (12.7.7) integral yig'indidan tashqari, *Darbuning quyi va yuqori yig'indilari* deb ataluvchi tushunchalar ham mavjuddir. Buning uchun $m_i = \inf_{x \in [x_{i-1}; x_i]} f(x)$

va $M_i = \sup_{x \in [x_{i-1}; x_i]} f(x)$ belgilashlar kiritib.

$$\bar{S}_n = \sum_{i=1}^n M_i \Delta x_i \quad (12.7.9)$$

va

$$\underline{S}_n = \sum_{i=1}^n m_i \Delta x_i \quad (12.7.10)$$

yig'indilarni tuzamiz. $\bar{S}_n$ - *Darbuning yuqori* $\underline{S}_n$ - esa *quyi yig'indisi* deb ataladi. (12.7.7), (12.7.9) va (12.7.10) larning tuzilishidan ko'rinadiki.

$$\underline{S}_n \leq \sigma_n \leq \bar{S}_n \quad (12.7.11)$$

o'rinlidir. (12.7.11)dan, Darbuning quyi va yuqori yig'indilari bitta chekli limitga ega bo'lsa, integral yig'indining ham shu limitga ega ekanligi, ya'ni aniq integral mavjud ekanligi kelib chiqadi. Bu o'rinda, $f(x)$ funksiya $[a, b]$ oraliqda integrallanuvchi, ya'ni aniq integrali mavjud bo'lishi uchun Darbuning quyi va yuqori yig'indilari bitta chekli limitga ega bo'lishi zarur va etarli bo'lishi isbotlanganligini aytamiz. Bundan tashqari aniq integralning mavjudligi haqidagi quyidagi teorema ham isbotlangandir [7] (uni isbotsiz keltiramiz).

12.7.1-teorema. Agar $f(x)$ funksiya $[a, b]$ kesmada uzluksiz bo'lsa, $\int_a^b f(x) dx$ mavjuddir.

Shuningdek, agar $f(x)$ funksiya $[a, b]$ kesmada bo'lakli uzluksiz, ya'ni chekli sondagi birinchi jins uzilish nuqtalari mavjud bo'lganda ham integrallanuvchi ekanligi isbotlangandir.

Bu o'rinda, (12.7.8) ga asosan aniq integral tatbiqi sifatida yuqorida ko'rilgan masalalarni keltirish mumkinligini aytamiz:

1) egri chizikli trapetsiyaning yuzi (12.7.2) dan $S = \int_a^b f(x)dx$;

2) to'g'ri chiziqli yo'l bo'ylab o'zgaruvchi kuchning bajargan ishi (12.7.4) dan $A = \int_a^b q(x)dx$;

3) kesma bo'ylab tarqalgan massa, (12.7.6) dan $m = \int_a^b \rho(x)dx$.

12.7.3. Aniq integralning xossalari

$$1^0. \int_a^b dx = b - a$$

2⁰. Agar $f(x)$ $[a;b]$ kesmada integrallanuvchi funksiya bo'lsa, $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3⁰. $f(a)$ mavjud bo'lsa, $\int_a^a f(x)dx = 0$

4⁰. Agar $f(x)$ $[a;b]$ kesmada integrallanuvchi funksiya bo'lsa, $\int_a^b Af(x)dx = A \int_a^b f(x)dx$, A - o'zgarmas

5⁰. Agar $f_i(x) (i = \overline{1, n})$ $[a;b]$ kesmada integrallanuvchi funksiyalar bo'lsa, $\int_a^b \left(\sum_{i=1}^n A_i f_i(x) \right) dx = \sum_{i=1}^n A_i \int_a^b f_i(x)dx$, A_i - o'zgarmaslar

6⁰. $a < b$, $\forall x \in [a, b] \Rightarrow \varphi(x) \leq f(x)$ va $f(x)$, $\varphi(x)$ funksiyalar $[a;b]$ kesmada integrallanuvchi bo'lsa, $\int_a^b \varphi(x)dx \leq \int_a^b f(x)dx$

7⁰. $c \in (a, b)$, agar $f(x)$ $[a;b]$ kesmada integrallanuvchi funksiya bo'lsa, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

8⁰. Agar $f(x)$ funksiya $[a, b]$ da uzluksiz, $q(x)$ esa ishorasini o'zgartirmovchi va $\int_a^b q(x)dx$ mavjud hamda $q(x)f(x)$ $[a;b]$ da integrallanuvchi bo'lsa, shunday $c \in [a, b]$

mavjudki, $\int_a^b f(x)q(x)dx = f(c) \int_a^b q(x)dx$ o'rinli bo'ladi. Bu xossani *o'rta qiymat haqidagi teorema* deb ham yuritiladi.

Bu xossalardan 1⁰, 4⁰, 7⁰- lari bevosita aniq integralning ta'rifi yordamida isbotlanadi, 2⁰- va 3⁰- lari aniq integral tushunchasini kengaytirish maqsadida qabul qilinadi. Oxirgisini isbotlaylik. $f(x)$ $[a, b]$ kesmada uzluksizligidan Veershtassning 2-teoremasiga ko'ra uning eng kichik m va eng katta M qiymatlari mavjud bo'lib,

$$\forall x \in [a, b], m \leq f(x) \leq M$$

o‘rinlidir $f(x)$ funksiya $[a,b]$ da o‘zgarmas bo‘lgan holda $m=M$ bo‘lib, xossa isboti 4^0 - xossaga asosan kelib chiqadi. Agar o‘zgaruvchi bo‘lsa, $m < M$ bo‘lib, $[a;b]$ da shunday x_1 va x_2 nuqtalar mavjud bo‘ladiki, $f(x_1) = m$, $f(x_2) = M$ bo‘ladi.

Endi, $q(x)$ ishorasini o‘zgartirmaydigan bo‘lgani uchun $\forall x \in [a,b]$ bo‘lganda yoki $q(x) \geq 0$ yoki $q(x) \leq 0$ bo‘ladi.

Aniqlik uchun $q(x) \geq 0$ deylik u holda 6^0 - xossaga ko‘ra $\int_a^b q(x)dx \geq 0$ bo‘ladi.

Hamda, $m q(x) \leq f(x) q(x) \leq M q(x)$ tengsizlik ham o‘rinlidir. Yana 4^0 - va 6^0 - xossalarga ko‘ra oxirgi tengsizlikdan

$$m \int_a^b q(x)dx \leq \int_a^b f(x)q(x)dx \leq M \int_a^b q(x)dx$$

ni olamiz. Agar $\int_a^b q(x)dx = 0$ bo‘lsa, oxirgidan teoremaning isboti o‘z-o‘zidan kelib

chiqadi. Agar $\int_a^b q(x)dx > 0$ bo‘lsa, oxirgi tengsizlikni unga bo‘lib,

$$m \leq \frac{\int_a^b f(x)q(x)dx}{\int_a^b q(x)dx} \leq M$$

ni olamiz. Endi,

$$\frac{\int_a^b f(x)q(x)dx}{\int_a^b q(x)dx} = \mu$$

deb belgilasak, bu μ son uchun $m \leq \mu \leq M$ bo‘lganligidan, agar $m = \mu$ bo‘lsa $c = x_1$ deb, $\mu = M$ bo‘lsa $c = x_2$ deb olamiz, $m < \mu < M$ bo‘lganda esa Koshining ikkinchi teoremasiga ko‘ra x_1 va x_2 lar orasida shunday c topiladiki, $f(c) = \mu$ bo‘ladi. Buni e‘tiborga olsak, oxirgi belgilashdagi μ o‘rniga $f(c)$ ni qo‘yib, xossa isbotiga kelamiz. $q(x) \leq 0$ bo‘lgan hol ham xuddi shunga o‘xshash qaraladi.

1-eslatma. 8^0 - xossada $q(x)f(x)$ ning $[a;b]$ da integrallanuvchi bo‘lishini talab qilish ortiqchadir, chunki $q(x)$ va $f(x)$ larning integrallanuvchi ekanligidan uning ko‘paytmasining integrallanuvchi bo‘lishi kelib chiqadi.

2-eslatma. Agar 8^0 - xossada $q(x) = 1$ desak, $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo‘lganda shunday $c \in (a,b)$ mavjud bo‘ladiki $\int_a^b f(x)dx = f(c)(b-a)$ o‘rinli bo‘ladi.

Bundan

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

bo‘lib, bu qiymatni $f(x)$ funksiyaning $[a,b]$ kesmadagi o‘rta qiymati deb yuritiladi.

3-eslatma. 4^0 –xossasida $A=0$ desak, $\int_a^b 0 dx = 0$ bo‘lishi kelib chiqadi.

12.7.4. Nyuton-Leybnis formulasi

Aytaylik, $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo‘lsin. U holda, $x \in [a,b]$ bo‘lganda $f(t)$ funksiya $[a,x]$ oraliqda integrallanuvchi bo‘ladi va

$$\Phi(x) = \int_a^x f(t) dt \quad (12.7.12)$$

funksiya $[a,b]$ kesmada aniqlangandir. Bu funksiyaning hosilasini topaylik. Buning uchun argument x ga $[a,b]$ kesmadan chiqmaydigan qilib Δx orttirma beramiz va $\Delta\phi$ funksiya orttirmasini hisoblaymiz:

$$\Delta\phi = \Phi(x + \Delta x) - \phi(x) = \int_a^{x+\Delta x} f(t) dt - \int_a^x f(t) dt$$

Oxirgi integrallardan birinchisiga 7^0 - xossani qo‘llab,

$$\Delta\phi = \int_a^x f(t) dt + \int_x^{x+\Delta x} f(t) dt - \int_a^x f(t) dt = \int_x^{x+\Delta x} f(t) dt$$

ni va bunga 8^0 – xossa, ya’ni o‘rta qiymat haqidagi teoremani qo‘llab,

$$\Delta\phi = f(c) \cdot (x + \Delta x - x) = f(c) \Delta x$$

ni olamiz. Bu yerda c x va $x + \Delta x$ orasidagi qiymatdir. Oxirgi tenglikdan

$$\frac{\Delta\phi}{\Delta x} = f(c)$$

kelib chiqadi. Bu tenglikda $\Delta x \rightarrow 0$ dagi limitga o‘tib va bunda $c \rightarrow x$ ekanligini e’tiborga olib, $f(x)$ funksiya uzluksizligidan

$$\phi'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta\phi}{\Delta x} = \lim_{c \rightarrow x} f(c) = f(x)$$

ni olamiz. Demak, (12.7.12) formula bilan kiritilgan $\phi(x)$ funksiyaning hosilasi $f(x)$ ga teng, ya’ni $y = f(x)$ ning boshlang‘ich funksiyasi ekan.

Faraz qilaylik, bizga $f(x)$ funksiyaning qandaydir $F(x)$ boshlang‘ich funksiyasi ma’lum bo‘lsin. U holda 12.1.1 –teoremaga asosan

$$\phi(x) = F(x) + C_0 \quad (12.7.13)$$

tenglik o‘rinlidir, bu yerda C_0 qandaydir o‘zgarmas son.

Agar (12.7.12) da $x=a$ desak, aniq integralning 3^0 - xossasi asosida $\phi(a)=0$ ga ega bo‘lamiz va (12.7.13) dan $F(a) + C_0 = 0 \Rightarrow C_0 = -F(a)$ bo‘lib, buni (12.7.13) ga qo‘yish natijasida $\phi(x) = F(x) - F(a)$ ga kelamiz. Oxirgida va (12.7.12) da $x=b$ deb,

$$\int_a^b f(x) dx = F(b) - F(a)$$

ni olamiz. Bu *Nyuton-Leybnis formulasidir*.

Bu formulaning ma’nosi shundan iboratki, agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo‘lsa, uning shu oraliqdagi boshlang‘ich funksiyasi mavjud bo‘lib, aniq integrali boshlang‘ich funksiyasining kesma bo‘yicha orttirmasiga tengdir.

Nyuton-Leybnis formulasini

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \quad (12.7.14)$$

ko‘rinishda ham yoziladi.

Misol: $\int_0^1 x dx = \frac{x^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} = 0,5.$

Demak, Nyuton-Leybnis formulasini integral osti funksiyasi integrallash oraliq‘ida uzluksiz bo‘lgan holda qo‘llash mumkin bo‘lib, u aniq integralni hisoblashni boshlang‘ich funksiyani topish va so‘ngra uning orttirmasini hisoblashga keltirar ekan.

Eslatma: Bu yerda $\phi'(x)=f(x)$ ekanligini ko‘rsatish bilan biror oraliqda uzluksiz bo‘lgan funksiya bu oraliqda boshlang‘ich funksiyaga ega ekanligini isbotladik.

12.7.5. Aniq integralda bo‘laklash usuli

Bu yerda $u(x)$ va $v(x)$ funksiyalar $[a,b]$ kesmada uzluksiz differensiallanuvchi deb faraz qilib,

$$d(uv) = vdu + u dv$$

ni $[a,b]$ bo‘yicha integrallasak, Nyuton-Leybnis formulasi asosida

$$\int_a^b u dv = (uv)\Big|_a^b - \int_a^b v du$$

ni olamiz. Bu *aniq integralda bo‘laklab integrallash usuli* deb yuritiladi.

Misol: $\int_0^1 x e^x dx = (x e^x)\Big|_0^1 - \int_0^1 e^x dx = 1 \cdot e^1 - 0 \cdot e^0 - e^x\Big|_0^1 = e - e + 1 = 1$

$|u = x, \quad dv = e^x dx, \quad du = dx, \quad v = e^x|.$

12.7.6. Aniq integralda o‘zgaruvchini almashtirish usuli

Agar $\int_a^b f(x)dx$ integralda ($f(x)$ funksiya $[a,b]$ kesmada uzluksiz) $x=\varphi(t)$

almashtirish qilingan bo‘lib, $\varphi(\alpha) = a$, $\varphi(\beta)=b$ va $\varphi(t)$ funksiya $[\alpha;\beta]$ (yoki $[\beta;\alpha]$) kesmada uzluksiz differensialanuvchi hamda $t \in [\alpha;\beta] \Rightarrow \varphi(t) \in a;b$ bo‘lsa,

$$\int_a^b f(x)dx = \int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)dt$$

o‘rinlidir. Haqiqatdan ham, $f(x)$ ning boshlang‘ich funksiyasini $F(x)$ desak,

$f[\varphi(t)]\varphi'(t)$ ning boshlang‘ich funksiyasi $F[\varphi(t)]$ dan iborat bo‘ladi hamda

$\int_a^b f(x)dx = F(b) - F(a)$, Nyuton – Leybnis formulasi asosida

$$\int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)dt = F[\varphi(\beta)] - F[\varphi(\alpha)] = F(b) - F(a)$$

bo‘lib, bulardan yuqoridagi tenglik kelib chiqadi.

Misol. $\int_0^a \sqrt{a^2 - x^2} dx$ ni hisoblang ($a \in R^+$)

Yechish: $x = a \sin t$ almashtirish qilamiz.

$$x = a \Rightarrow t = \frac{\pi}{2}, \quad x = 0 \Rightarrow t = 0, \quad dx = a \cos t dt, \quad t \in \left[0; \frac{\pi}{2}\right] \Rightarrow x \in [a; b];$$

$$\begin{aligned} \int_0^a \sqrt{a^2 - x^2} dx &= \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 t} \cdot a \cos t dt = a^2 \int_0^{\pi/2} \cos^2 t dt = \\ &= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2t) dt = \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) \Big|_0^{\pi/2} = \\ &= \frac{a^2}{2} \left(\left(\frac{\pi}{2} + \frac{1}{2} \sin \pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right) = \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} a^2. \end{aligned}$$

Bu misoldan ko‘rinadiki, aniq integralda o‘zgaruvchi almashtirilganda aniqmas integraldagi kabi eski o‘zgaruvchiga qaytish shart emas ekan.

Endi juft va toq funksiyalarning simmetrik oraliq bo‘yicha integralini ko‘raylik.

Aytaylik, $f(x)$ $[-a, a]$ simmetrik kesmada aniqlangan va integrallanuvchi bo‘lsin.

Quyidagi $\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$ tenglik o‘rinli ekanligini bilamiz. Bu

tenglikning o‘ng tomonidagi birinchi integralda $x = -t$ almashtirish qilib,

$$\int_{-a}^0 f(x) dx = - \int_a^0 f(-t) dt = \int_0^a f(-x) dx \quad \text{ni olamiz va buni hisobga olib,}$$

$$\int_{-a}^a f(x) dx = \int_0^a (f(-x) + f(x)) dx \quad \text{ga ega bo‘lamiz.}$$

Bundan: a) $f(x)$ juft funksiya bo‘lsa,

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad (12.7.15)$$

o‘rinli ekanligiga ishonch hosil qilamiz;

b) agar $f(x)$ toq funksiya bo‘lsa,

$$\int_{-a}^a f(x) dx = 0 \quad (12.7.16)$$

ekanligi kelib chiqadi.

Demak, agar $f(x)$ funksiya $[-a; a]$ simmetrik oraliqda aniqlangan va integrallanuvchi bo‘lib, juft bo‘lsa (12.7.15) tenglik toq bo‘lganda esa (12.7.16) tenglik o‘rinlidir.

12.7.7. Aniq integralning geometriyaga tatbiqlari

1. Tekis shakl yuzini hisoblash. a) Yuqorida egri chiziqli trapetsiyaning yuzini hisoblash uchun

$$S = \int_a^b f(x) dx \quad (12.7.15)$$

formulani chiqardik. Bu yerda $f(x)$ $[a;b]$ kesmada uzluksiz va manfiy bo'lmagan funksiyadir (12.7.3 a - rasm).

b) Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz va musbat bo'lmasa,

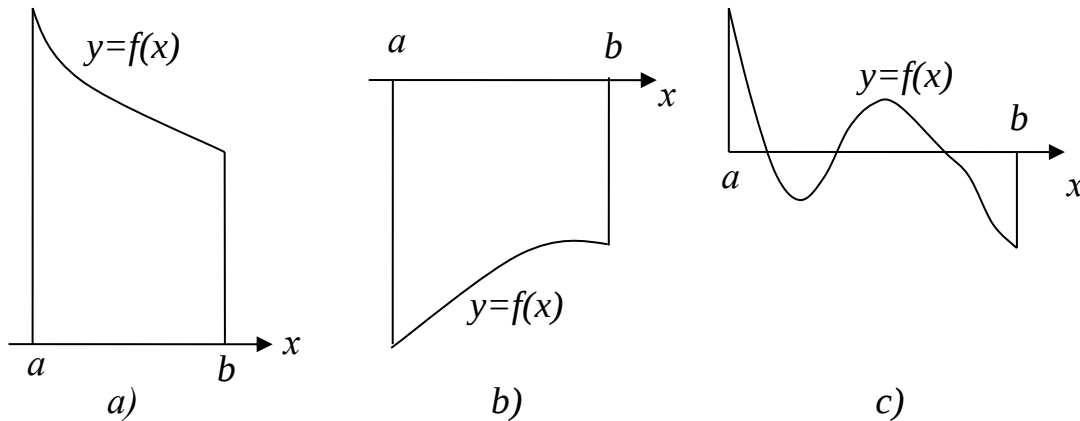
$$S = -\int_a^b f(x)dx \quad (12.7.16)$$

formula egri chiziqli trapetsiya yuzining qiymatini to'g'ri beradi (12.7.3b-rasm) (agar «-» ishora integral oldiga qo'yilmasa, yuza qiymati manfiy bo'lib qolar edi).

c) Agar $f(x)$ funksiya $[a,b]$ oraliqda uzluksiz va ishorasini o'zgartirsa (12.7.3 c-rasm), u holda egri chiziqli trapetsiya yuzining qiymati uchun

$$S = \int_a^b |f(x)| dx \quad (12.7.17)$$

formula to'g'ridir.

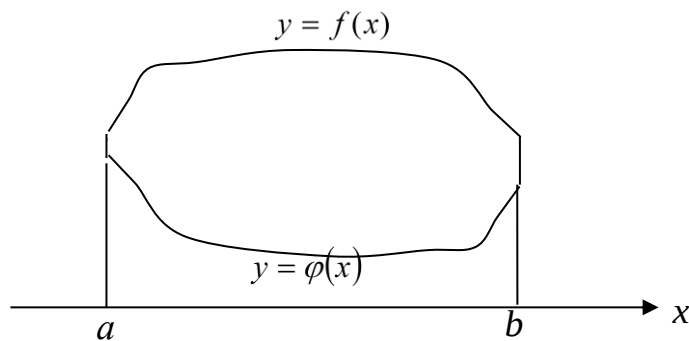


12.7.3 – rasm.

d) Tekis shakl yuqoridan $y=f(x)$ uzluksiz funksiya, quyidan esa $y=\varphi(x)$ uzluksiz funksiya grafiklari bilan $[a;b]$ kesmada chegaralangan bo'lsa, uning yuzi uchun

$$S = \int_a^b [f(x) - \varphi(x)]dx \quad (12.7.18)$$

formula o'rinlidir, bu yerda $\forall x \in [a;b], \varphi(x) \leq f(x)$ (12.7.4-rasm).

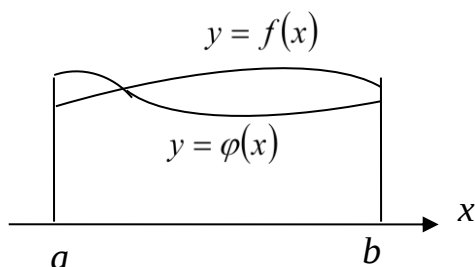


12.7.4-rasm.

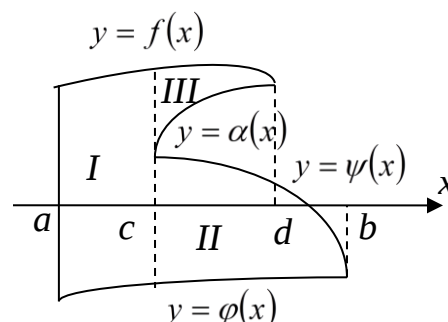
e) Agar tekis shakl $[a;b]$ kesmada $y=f(x)$ va $y=\varphi(x)$ uzluksiz funksiyalarning garfiklari bilan chegaralangan bo‘lib, ular kesishsa, bu shakl yuzi uchun

$$S = \int_a^b |f(x) - \varphi(x)| dx \quad (12.7.19)$$

formula o‘rinlidir (12.7.5-rasm).



12.7.5 –rasm.



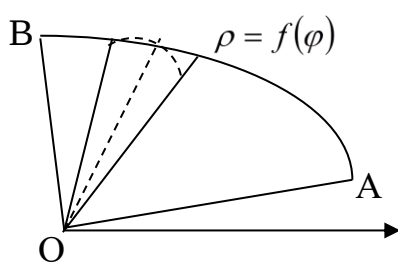
12.7.6 –rasm.

f) Agar tekis shakl murakkabroq bo‘lib, yuqoridagi hollardan birortasiga ham to‘g‘ri kelmasa, uni bo‘laklarga shunday ajratish kerakki, har bir bo‘lakka yuqoridagi formulalardan biri to‘g‘ri kelsin. Masalan, 12.7.6-rasmdagi shaklni qarasak, uni uchta I, II va III bo‘laklarga ajratilsa, 12.7.6-rasmdan ko‘rinadiki,

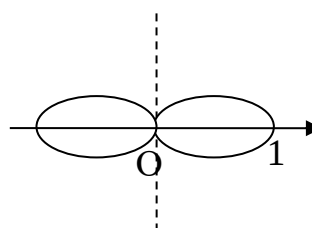
$$S_I = \int_a^c [f(x) - \varphi(x)] dx, \quad S_{II} = \int_c^d [\psi(x) - \varphi(x)] dx, \quad S_{III} = \int_c^d [f(x) - \alpha(x)] dx$$

larni hisoblab, tekis shakl yuzi uchun $S=S_I+S_{II}+S_{III}$ ni olamiz.

m) **Qutb koordinatalar tekisligida berilgan tekis shakl yuzini hisoblash.** Qutb koordinatalar sistemasida $\rho=f(\varphi)$, $\varphi \in [\alpha; \beta]$ uzluksiz funksiya berilgan bo‘lsa, uning grafigi va chetki qutb radiuslari (12.7.7-rasmda OA va OB lar) bilan chegaralangan shakl *egri chiziqli sektor* deb ataladi. Xususiyl holda $\rho=f(\varphi)$ grafigi aylana yoyidan (ya‘ni $\rho=R$ -o‘zgarmas) iborat bo‘lsa, u doiraviy sektor bo‘ladi.



12.7.7 –rasm.



12.7.8 –rasm.

Bunday egri chiziqli sektor yuzini hisoblash masalasini qo‘yib, $[\alpha; \beta]$ kesmani ixtiyoriy qilib, n ta bo‘laklarga $\alpha=\varphi_0 < \varphi_1 < \dots < \varphi_n=\beta$ bo‘lib har bir bo‘linish nuqtasining qutb radiuslarini o‘tkazsak, egri chiziqli sektor n ta bo‘laklarga bo‘linadi. Bu bo‘laklardan k -siga mos keluvchi $[\varphi_{k-1}; \varphi_k]$ kesmaga tegishli ψ_k ni olib, $f(\psi_k)$ ni hisoblab, bu bo‘lak yuzi ΔS_k ning taqribiy qiymati sifatida radiusi $f(\psi_k)$ ga markaziy burchagi $\Delta\varphi_k = \varphi_k - \varphi_{k-1}$ ga teng bo‘lgan doiraviy sektor yuzini qabul qilamiz (12.7.7-rasmga qarang). U

holda, bu ishni barcha bo‘laklar uchun bajargach, egri chiziqli sektor yuzi S ning taqribiy qiymati uchun

$$S \approx \sum_{k=1}^n \frac{1}{2} f^2(\psi_k) \cdot \Delta\varphi_k$$

ga ega bo‘lamiz. Bu taqribiy tenglikning o‘ng tomonidagi ifoda $\frac{1}{2} f^2(\varphi)$ funksiyaning $[\alpha; \beta]$ kesma bo‘yicha integral yig‘indisi ekanligidan $\lambda = \max_k \Delta\varphi_k \rightarrow 0$ dagi limitga o‘tish natijasida

$$S = \frac{1}{2} \int_{\alpha}^{\beta} f^2(\varphi) d\varphi$$

ni olamiz. Bu *egri chiziqli sektorning yuzini hisoblash formulasidir*.

1-misol. Qutb koordinatalari sistemasida $\rho = |\cos\varphi|$ chiziq bilan chegaralangan shakl yuzi hisoblansin (12.7.8-rasm).

Yechish. $f(\varphi) = |\cos\varphi|$, $\alpha = 0$, $\beta = 2\pi$.

$$S = \frac{1}{2} \int_0^{2\pi} \cos^2 \varphi d\varphi = \frac{1}{4} \int_0^{2\pi} (1 + \cos 2\varphi) d\varphi = \frac{1}{4} \left(\int_0^{2\pi} d\varphi + \int_0^{2\pi} \cos 2\varphi d\varphi \right) = \frac{1}{4} \left(2\pi + \frac{1}{2} \sin 2\pi \Big|_0^{2\pi} \right) = \frac{\pi}{2}.$$

2. Parallel kesimlarining yuzi bo‘yicha jismning hajmini hisoblash.

Aytaylik, jism Ox o‘qi yo‘nalishi bo‘yicha $[a; b]$ kesmaga joylashgan bo‘lib, $\forall x \in [a; b]$ nuqtada uning absissalar o‘qiga perpendikulyar kesimining yuzi $S(x)$ uzluksiz funksiya orqali ifodalansin (12.7.9-rasmga qarang), ya’ni uning parallel kesimlarining yuzi har bir x uchun ma’lum bo‘lsin. Uning hajmini hisoblash uchun $[a; b]$ ni ixtiyoriy qilib n ta bo‘laklarga bo‘lamiz va i -bo‘lakka mos keluvchi x_{i-1} va x_i nuqtalar orqali o‘tkazilgan Ox o‘qqa perpendikulyar kesimlar orasida qolgan qismini asosi $S(\xi_i)$ (bu yerda $\xi_i \in [x_{i-1}; x_i]$) va balandligi Δx_i bo‘lgan silindr bilan almashtirib, bu bo‘lakcha hajmining taqribiy qiymati uchun

$$\Delta v_i \approx S(\xi_i) \Delta x_i$$

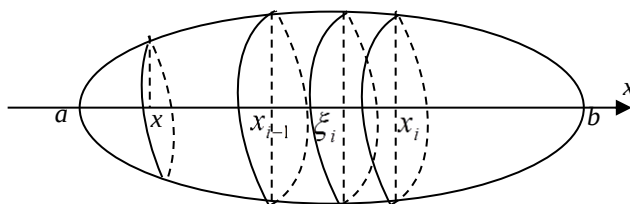
ni qabul qilamiz. Bu ishni barcha bo‘lakchalar uchun bajarib, jism hajmining taqribiy qiymati uchun

$$V \approx \sum_{i=1}^n S(\xi_i) \Delta x_i$$

ga ega bo‘lamiz. Bu taqribiy formulaning o‘ng qismi $S(x)$ funksiyaning $[a; b]$ oraliq bo‘yicha integral yig‘indisidir. Demak, $\lambda = \max_i \Delta x_i \rightarrow 0$ dagi limitga o‘tish natijasida

$$V = \int_a^b S(x) dx \quad (12.7.20)$$

ni olamiz. Bu *parallel kesimlarining yuzi ma’lum bo‘lganda jismning hajmini hisoblash formulasidir*. $S(x)$ funksiyaning $[a; b]$ kesmada uzluksiz deb faraz qilamiz.



12.7.9 –rasm.

3. Aylanish jismining hajmi. Aytaylik, $[a; b]$ kesmada manfiy bo'lmagan uzluksiz $y=f(x)$ funksiya grafigi, $x=a$, $x=b$, $y=0$ to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiya Ox o'qi atrofida aylanishidan jism hosil bo'lgan bo'lsin. Shu aylanish jismining hajmini topaylik (12.7.10-rasm)

Agar $x \in [a; b]$ nuqtadagi Ox o'qiga perpendikulyar kesimni qarasak, y radiusi $y=f(x)$ bo'lgan doiradan iboratdir ya'ni uning yuzi uchun $S(x) = \pi y^2 = \pi f^2(x)$ ni olamiz. Buni (12.7.20) ga qo'yib,

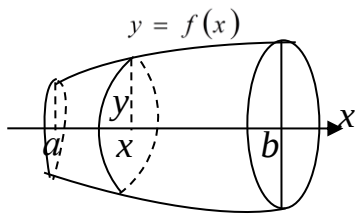
$$V = \pi \int_a^b y^2 dx = \pi \int_a^b f^2(x) dx \quad (12.7.21)$$

ga ega bo'lamiz. Bu yuqorida aytilgan aylanish jismi hajmining formulasidir.

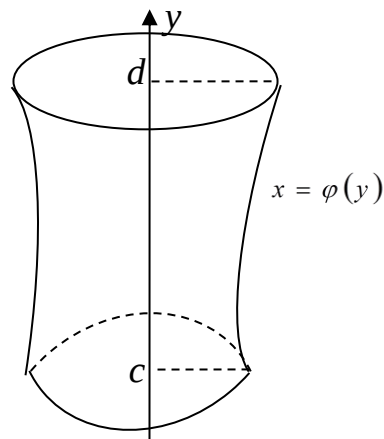
Xuddi shunga o'xshash, $x=\varphi(y)$ grafigi $y=c$, $y=d$, $x=0$ chiziqlar bilan chegaralangan egri chiziqli trapetsiya ($c < d$) Oy o'qi atrofida aylantirilsa, aylanish jismining hajmi uchun

$$V = \pi \int_c^d x^2 dy = \pi \int_c^d \varphi^2(y) dy \quad (12.7.22)$$

formulaga ega bo'lamiz (12.7.11-rasm)



12.7.10 –rasm.



12.7.11 –rasm.

2-misol. $y = \sqrt{x}$, $x=0$, $x=1$, $y=0$ chiziqlar bilan chegaralangan egri chiziqli trapetsiyani Ox o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi hisoblansin.

Yechish. $V = \pi \int_0^1 y^2 dx = \pi \int_0^1 (\sqrt{x})^2 dx = \pi \int_0^1 x dx = \pi \cdot \frac{x^2}{2} \Big|_0^1 = \pi \cdot \frac{1}{2} = \frac{\pi}{2};$

$$V = \frac{\pi}{2}.$$

4. Aylanish sirtining yuzi. Aytaylik, $[a; b]$ kesmada manfiy bo'lmagan uzluksiz differensiallanuvchi $y=f(x)$ funksiya grafigini Ox o'qi atrofida aylantirish natijasida sirt hosil qilingan bo'lsin (12.7.12-rasm). Shu aylanish sirtining yuzini hisoblash masalasini qo'yaylik. Bu masalani yechish uchun $[a; b]$ kesmani ixtiyoriy n ta bo'laklarga bo'lamiz va bo'linish nuqtalari orqali Ox o'qiga perpendikulyar tekisliklar o'tkazib sirtni bo'laklarga ajratamiz, i -bo'lak $[x_{i-1}; x_i]$ ($i=1; 2; \dots; n$) ga mos keluvchi sirtning qismini

12.7.12-rasmdagidek kesik konus yon sirti bilan almashtiramiz. U holda, kesik konus yon sirtining yuzining formulasiga ko'ra

$$\begin{aligned}\Delta S_i &\approx 2\pi \frac{y_{i-1} + y_i}{2} \Delta p_i = 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \\ &= 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i = 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{1 + (f'(\xi))^2} \Delta x_i.\end{aligned}$$

ni olamiz, bu yerda ΔS_i aylanish sirti i-bo'lagining yuzi, $\xi_i \in [a; b]$ qandaydir nuqta. Endi yuqolridagi taqribiy formulani integral yig'indiga moslash maqsadida quyidagi ishlarni bajaramiz:

$$\begin{aligned}\left| \frac{1}{2} [f(x_{i-1}) + f(x_i)] - f(\xi_i) \right| &= \frac{1}{2} |[f(x_{i-1}) - f(\xi_i)] + [f(x_i) - f(\xi_i)]| = \\ &= \frac{1}{2} |[f(x_i) - f(\xi_i)] - [f(\xi_i) - f(x_i)]| = \frac{1}{2} |f'(\lambda_i)(x_i - \xi_i) - f'(\mu_i)(\xi_i - x_{i-1})| \leq \\ &\leq \frac{1}{2} (|f'(\lambda_i)|(x_i - \xi_i) + |f'(\mu_i)|(\xi_i - x_{i-1})) \leq \frac{M_1}{2} (x_i - \xi_i + \xi_i - x_{i-1}) = \frac{M_1}{2} \Delta x_i\end{aligned}$$

bu yerda $M_1 = \max_{x \in [a; b]} |f'(x)|$. Oxiridan ko'rinadiki, $\frac{1}{2} [f(x_{i-1}) + f(x_i)]$ bilan $f(\xi_i)$ larning farqi $\lambda = \max_i \Delta x_i \rightarrow 0$ da cheksiz kichik miqdor ekan, shu sababli

$$\Delta S_i \approx 2\pi f(\xi_i) \sqrt{1 + [f'(\xi_i)]^2} \Delta x_i$$

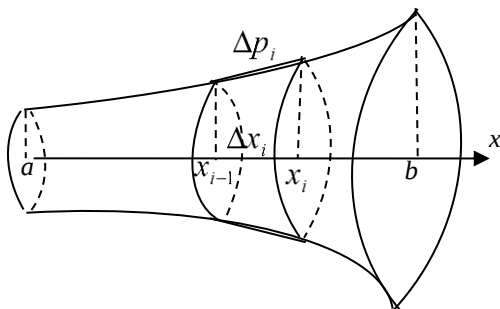
deb qabul qilish mumkin. U vaqtda aylanish sirtining yuzi S uchun

$$S \approx \sum_{i=1}^n 2\pi f(\xi_i) \sqrt{1 + [f'(\xi_i)]^2} \Delta x_i$$

taqribiy formulaga ega bo'lamiz va uning o'ng tomoni $2\pi f(x) \cdot \sqrt{1 + [f'(x)]^2}$ funksiyaning $[a; b]$ kesma bo'yicha integral yig'indisi ekanligidan

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2} dx \quad (12.7.23)$$

ga ega bo'lamiz. Bu aylanish sirtining yuzini topish formulasidir.



12.7.12-rasm.

3-misol. $y = \sqrt{x}$ funksiya grafigining $x \in [1; 2]$ qismini Ox o'qi atrofida aylantirishdan hosil bo'lgan sirtning yuzini hisoblang.

Yechish. $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$. Bularni (12.7.23) ga qo'yamiz.

$$\begin{aligned}
S &= 2\pi \int_1^2 \sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx = 2\pi \int_1^2 \sqrt{x + \frac{1}{4}} dx = \\
&= 2\pi \cdot \frac{2}{3} \left(\sqrt{x + \frac{1}{4}} \right)^3 \Big|_1^2 = \frac{4\pi}{3} \cdot \left(\left(\sqrt{2 + \frac{1}{4}} \right)^3 - \left(\sqrt{1 + \frac{1}{4}} \right)^3 \right) = \\
&= \frac{4\pi}{3} \left(\frac{27}{8} - \frac{5\sqrt{5}}{8} \right) = \frac{27 - 5\sqrt{5}}{6} \pi \approx 8,283.
\end{aligned}$$

5.Yoyning uzunligini hisoblash. Aytaylik, Dekart koordinatalar sistemasida t parametrning $[t_A; t_B]$ kesmada monoton o'zgarishga mos $x = \varphi(t)$, $y = \psi(t)$ parametrik tenglamalar bilan aniqlanuvchi chiziqning AB yoyi berilgan bo'lsin. Aytilgan kesmada $\varphi(t)$ va $\psi(t)$ funksiyalar uzluksiz differensiallanuvchi deb faraz qilamiz. Endi, $[t_A; t_B]$ kesmani ixtiyoriyicha qilib n ta bo'laklarga ajratamiz va bo'linish nuqtalarini o'sish tartibida $t_A = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = t_B$ kabi belgilaymiz. Bu nuqtalardan k -siga AB yoyning $M_k(x_k; y_k)$ nuqtasi mos keladi, bu yerda $x_k = \varphi(t_k)$, $y_k = \psi(t_k)$, $k = 0; 1; \dots; n$. Bu nuqtalarni ketma-ket tutashtirish natijasida AB yoyga ichki chizilgan siniq chiziqqa ega bo'lamiz va uning uzunligini c_n bilan belgilaymiz.

Agar yoyga ichki chizilgan siniq chiziq uzunligining siniq chiziq tomonlari cheksiz kichiklashishining natijasidagi chekli limiti mavjud bo'lsa, va u siniq chiziq ichki uchlarining qanday olinishiga bog'liq bo'lmasa, bu limit yoy uzunligi deb qabul qilinadi. Bu yoy uzunligining ta'rifidir.

Endi, 12.7.13-rasmdagidek Δp_k bilan siniq chiziq k - tomonining uzunligini belgilasak, $\Delta p_k = \sqrt{\Delta x_k^2 + \Delta y_k^2}$ bo'lishi ayondir. U holda, siniq chiziqning uzunligi

$c_n = \sum_{k=1}^n \sqrt{\Delta x_k^2 + \Delta y_k^2}$ dan iboratdir. Agar $\varphi'(t)$ va $\psi'(t)$ –uzluksiz hosilalar mavjudligini e'tiborga olsak, chekli orttirmalar formulasiga ko'ra

$\Delta x_k = \varphi'(\tau_k) \Delta t_k$, $\Delta y_k = \psi'(\theta_k) \Delta t_k$ larni olamiz, bu yerda τ_k va θ_k lar $[t_{k-1}; t_k]$ ga tegishli qandaydir nuqtalardir. Bularni hisobga olib,

$$C_n = \sum_{k=1}^n \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} \Delta t_k$$

ga ega bo'lamiz. Buni

$$C_n = \sum_{k=1}^n \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} \Delta t_k + \sum_{k=1}^n \left[\sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} - \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\tau_k))^2} \right] \Delta t_k$$

ko'rinishda yozib olsak, o'ng tomondagi yig'indining birinchi ko'shiluvchisi $\sqrt{(\varphi'(t))^2 + (\psi'(t))^2}$ funksiyaning $[t_A; t_B]$ oraliq bo'yicha integral yig'indisidan iboratdir. Ikkinchisini baholaylik. Uni R_n deb belgilab olamiz:

$$\begin{aligned}
R_n &= \sum_{k=1}^n \left[\sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} - \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\tau_k))^2} \right] \Delta t_k = \\
&= \sum_{k=1}^n \frac{(\psi'(\theta_k))^2 - (\psi'(\tau_k))^2}{\sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} + \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\tau_k))^2}} \Delta t_k =
\end{aligned}$$

$$= \sum_{k=1}^n \frac{(\psi'(\theta_k) - \psi'(\tau_k))(\psi'(\theta_k) + \psi'(\tau_k))}{\sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} + \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\tau_k))^2}} \Delta t_k.$$

$\psi'(t)$ uzluksizligidan $\max \Delta t_k \rightarrow 0$ da $\psi'(\theta_k) - \psi'(\tau_k) = \alpha_k$ cheksiz kichik miqdor, ya'ni $\alpha_k \rightarrow 0$ ($k=1;2;\dots;n$) ekanligi, bundan esa $\max_k |\alpha_k| \rightarrow 0$ kelib chiqadi.

Undan tashqari,

$$0 \leq \frac{|\psi'(\theta_k) + \psi'(\tau_k)|}{\sqrt{(\varphi'(\tau_k))^2 + (\psi'(\theta_k))^2} + \sqrt{(\varphi'(\tau_k))^2 + (\psi'(\tau_k))^2}} \leq 1$$

ni ko'rsatish qiyin emas. U holda

$$|R_n| \leq \sum_{k=1}^n |\alpha_k| \cdot \Delta t_k \leq (t_B - t_A) \max_k |\alpha_k| \rightarrow 0,$$

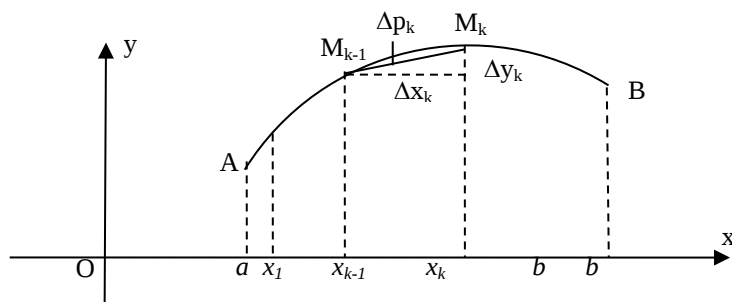
ya'ni R_n cheksiz kichik miqdor ekanligi kelib chiqadi.

Shunday qilib, $\lim_{\max \Delta t_k \rightarrow 0} C_n = \int_{t_A}^{t_B} \sqrt{(\varphi'(t))^2 + (\psi'(t))^2} dt$ chekli limitning mavjudligini va

yoy uzunligi ta'rifiga ko'ra AB yoy uzunligi l uchun

$$l = \int_{t_A}^{t_B} \sqrt{(\varphi'(t))^2 + (\psi'(t))^2} dt \quad (12.7.24)$$

formulani olamiz.



12.7.13-rasm.

Agar AB yoy x argumentning $[a;b]$ kesmada monoton o'zgarishiga (o'sishiga) mos keluvchi $y=f(x)$ funksiyaning grafigi sifatida berilgan bo'lib, bu kesmada uzluksiz $f'(x)$ hosila mavjud bo'lsa, (12.7.24) ni

$$l = \int_a^b \sqrt{1 + (f'(x))^2} dx \quad (12.7.25)$$

ko'rinishda yozish mumkin.

AB yoy qutb koordinatalari tekisligida $\varphi \in [\alpha; \beta]$, $\rho = f(\varphi)$ tenglama bilan berilgan bo'lsa, t parametr sifatida φ ni qabul qilib nuqtaning Dekart va qutb koordinatalari orasidagi $\begin{cases} x = \rho \cos \varphi, \\ y = \rho \sin \varphi \end{cases}$ bog'lanishni eslasak, (12.7.24) dan

$$l = \int_{\alpha}^{\beta} \sqrt{\rho^2 + (\rho')^2} d\varphi \quad (12.7.26)$$

formulani olamiz. Bu yerda $f(\varphi) - [\alpha; \beta]$ kesmada uzluksiz differensiallanuvchi deb faraz qilindi.

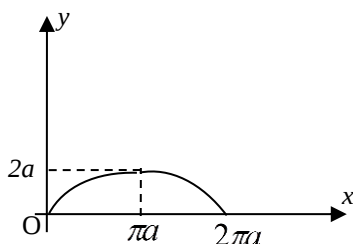
Eslatma. Agar egri chiziqlik yuqoridagi ma'noda uzunlikka ega bo'lsa, uni *to'g'rilanuvchi* deb ataydilar va undan tashqari, bu vaqtda yoyning ΔS bo'lagini tortib turuvchi Δp vatar nolga intilganda $\lim_{\Delta p \rightarrow 0} \frac{\Delta S}{\Delta p} = 1$, ya'ni $\Delta S \sim \Delta p$ bo'lishi isbotlangandir.

4-misol. $x=a(t-\sin t)$, $y=a(1-\cos t)$ ($a>0$) sikloida bitta arkining ($0 \leq t \leq 2\pi$) uzunligini hisoblang (12.7.14- rasm).

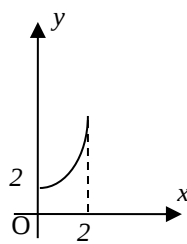
Yechish. $x'=a(1-\cos t)$, $y'=a\sin t$

$$l = \int_0^{2\pi} \sqrt{[a(1-\cos t)]^2 + (a\sin t)^2} dt = a \int_0^{2\pi} \sqrt{1-2\cos t + \cos^2 t + \sin^2 t} dt =$$

$$= a \int_0^{2\pi} \sqrt{2(1-\cos t)} dt = a \int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}} dt = 2a \int_0^{2\pi} \sin \frac{t}{2} dt = 2a \left(-2\cos \frac{t}{2} \right) \Big|_0^{2\pi} = -4a(\cos \pi - \cos 0) = 8a$$



12.7.14 rasm.



12.7.15 rasm.

5-misol. $y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$ zanjir chizig'ining (12.7.15- rasm) (0;2) uchidan abssissasi 2 ga teng bo'lgan nuqtasigacha bo'lgan qismi yoyining uzunligini hisoblang.

Yechish: $y' = \frac{1}{2} \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}} \right)$;

$$l = \int_0^2 \sqrt{1 + \left(\frac{1}{2} \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}} \right) \right)^2} dx = \frac{1}{2} \int_0^2 \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}} \right) dx =$$

$$= \frac{1}{2} \cdot 2 \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}} \right) \Big|_0^2 = e - e^{-1} - (e^0 - e^0) = e - e^{-1} = e - \frac{1}{e}.$$

6-misol. Qutb koordinatalar tekisligida berilgan $\rho=2a(1+\cos\varphi)$ kordioda (12.7.16- rasm) yoyining uzunligini hisoblang.

Yechish: $0 \leq \varphi \leq 2\pi$, $\rho' = 2a(0 - \sin \varphi) = -2a \sin \varphi$

$$l = \int_0^{2\pi} \sqrt{(2a(1+\cos\varphi))^2 + (-2a \sin \varphi)^2} d\varphi = 2a \int_0^{2\pi} \sqrt{1+2\cos\varphi + \cos^2 \varphi + \sin^2 \varphi} d\varphi =$$

$$= 2\sqrt{2}a \int_0^{2\pi} \sqrt{1+\cos\varphi} d\varphi = 2\sqrt{2}a \int_0^{2\pi} \sqrt{2\cos^2 \frac{\varphi}{2}} d\varphi = 4a \int_0^{2\pi} \cos \frac{\varphi}{2} d\varphi = 8a \int_0^{\pi} \cos \frac{\varphi}{2} d\varphi = 8a \left(2 \sin \frac{\varphi}{2} \right) \Big|_0^{\pi} = 16a.$$

Agar AB fazoviy egri chiziqlik $t \in [t_A; t_B]$ $x=\varphi(t)$, $y=\psi(t)$, $z=\phi(t)$ parametrik tenglamalari bilan berilgan bo'lsa, tekis egri chiziqlik uchun qilingan ishlarni takrorlab, uning uzunligi uchun

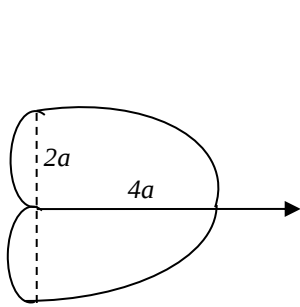
$$l = \int_{t_A}^{t_B} \sqrt{(x')^2 + (y')^2 + (z')^2} dt \quad (12.7.27)$$

formulani olamiz. Bu yerda $\varphi(t), \psi(t), \phi(t)$ funksiyalar uzluksiz differensiallanuvchi deb faraz qilindi.

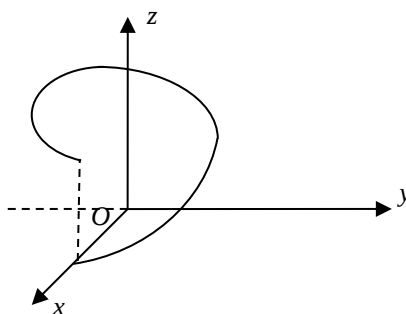
7-misol. $x = acost, y = asint, z = bt$ ($a > 0, b > 0$) vint chizig'ining (12.7.17-rasm) bir qadamiga tegishli qismining uzunligini hisoblang.

Yechish. $0 \leq t \leq 2\pi; \quad x' = -a \sin t, \quad y' = a \cos t, \quad z' = b$

$$l = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2 + b^2} dt = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi \sqrt{a^2 + b^2}.$$



12.7.16 rasm.



12.7.17 rasm.

Agar AB yoyining AC qismining uzunligi l ni qarasak, bu yerda C nuqta yoyining A va B uchlari orasidagi qo'zg'aluvchi nuqta (12.7.18_{a,b}- rasm), u vaqtda yuqorida chiqarilgan formularni quyidagicha yozish mumkin:

$$l = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2} d\tau, \quad t \in [t_A; t_B]$$

$$l = \int_0^x \sqrt{1 + (f'(\xi))^2} d\xi, \quad x \in [a; b]$$

$$l = \int_0^\varphi \sqrt{\rho^2(\psi) + (\rho'(\psi))^2} d\psi, \quad \varphi \in [\alpha; \beta]$$

$$l = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2} d\tau, \quad t \in [t_A; t_B]$$

Bularni differensiallab, yoy uzunligining differensial uchun quyidagi

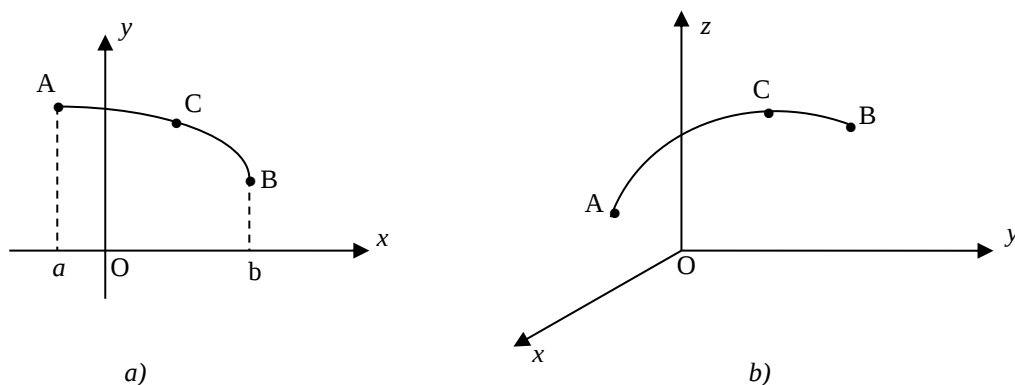
$$dl = \sqrt{dx^2 + dy^2}$$

$$dl = \sqrt{1 + (f'(x))^2} dx$$

$$dl = \sqrt{\rho^2(\varphi) + (\rho'(\varphi))^2} d\varphi$$

$$dl = \sqrt{dx^2 + dy^2 + dz^2}$$

formulalarga ega bo'lamiz. Bulardan so'ngisi fazoviy egri chiziq uchundir.



12.7.18 rasm

12.7.8. Aniq integralning mexanikaga tatbiqlari

Yuqorida, to'g'ri chiziqli yo'lda o'zgaruvchi kuchning bajargan ishini, o'zgaruvchi zichlikka ega bo'lgan sterjen (tayoqcha) massasini hisoblash masalalarini qarab, ularni aniq integral orqali ifodalagan edik. Ya'ni o'zgaruvchi $F(x)$ kuchning $[a;b]$ kesmadan iborat yo'lda bajargan ishi uchun

$$A = \int_a^b F(x) dx$$

formulani, o'zgaruvchi $\rho(x)$ zichlikka ega bo'lgan $[a;b]$ kesmadan iborat sterjen massasi uchun esa

$$m = \int_a^b \rho(x) dx$$

formulani oldik. Endi mexanikaning ba'zi bir masalalarini aniq integral yordamida yechish misollarini keltiramiz.

1.Og'irlik markazi. Aytaylik, n ta $A_1, A_2, \dots, A_n$ moddiy nuqtalarning qandaydir sistemasi berilgan bo'lib, ularning massalari mos ravishda $m_1, m_2, \dots, m_n$ bo'lsin. Agar bu moddiy nuqtalarning barchasi bitta tekislikka joylashgan bo'lsa, bu tekislikda kiritilgan Dekart koordinatalar sistemasida $A_i(x_i, y_i)$ ($i=1, 2, \dots, n$) bo'lib, bu moddiy nuqtalar sistemasining og'irlik markazini $C(x, y)$ desak, mexanikadan ma'lum

$$x = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}, \quad y = \frac{\sum_{i=1}^n m_i y_i}{\sum_{i=1}^n m_i} \quad (12.7.28)$$

formular o'rinalidir.

Agar tekis egri chiziq yoki tekis shakl qaralayotgan bo'lsa, (12.7.28) formulalar yaroqsizdir.

a) Tekis egri chiziqning og'irlik markazi. Faraz qilaylik, tekis to'g'rilanuvchi AB yoy (12.7.19-rasmga qarang), o'zining

$$x=x(s), \quad y=y(s), \quad s \in [0; S] \quad (12.7.29)$$

parametrik tenglamalari bilan berilgan bo'lib, parametr s sifatida A nuqtasidan boshlab hisoblangan qaralayotgan $C(x, y)$ nuqtasigacha bo'lgan yoy bo'lagining uzunligi undan tashqari, bu nuqtadagi yoy zichligi $\rho(s)$ dan iborat deb qabul qilingan bo'lsin. Agar AB

yoy uzunligini S bilan belgilasak, $s \in [0;S]$ bo'lishi ravshandir. $[0;S]$ kesmani (ya'ni AB yoyni) ixtiyoriy tanlangan

$$0=s_0 < s_1 < \dots < s_{i-1} < s_i < \dots < s_{n-1} < s_n = S$$

tugun nuqtalari yordamida, n ta bo'laklarga ajratamiz va i - bo'lakdan AB yoyda $A_i(x(s'_i); y(s'_i))$, bu yerda $s'_i \in [s_{i-1}; s_i]$, nuqtani olib, bu bo'lakcha birjinsli va uning zichligi $\rho(s'_i)$ ga teng hamda uning massasi $A_i (i=\overline{1;n})$ nuqtada mujassamlangan deb faraz qilamiz. Bu vaqtda AB yoyni taqriban n ta massasi $m_i = \rho(s'_i)\Delta s_i$ bo'lgan

$A_i (i=\overline{1;n})$ moddiy nuqtalar sistemasi bilan almashtirsak, bu sistemaning og'irlik markazi $(\bar{x}; \bar{y})$ uchun (12.7.28) formula asosida

$$\bar{x} = \frac{\sum_{i=1}^n x(s'_i)\rho(s'_i)\Delta s_i}{\sum_{i=1}^n \rho(s'_i)\Delta s_i}, \quad \bar{y} = \frac{\sum_{i=1}^n y(s'_i)\rho(s'_i)\Delta s_i}{\sum_{i=1}^n \rho(s'_i)\Delta s_i}$$

ni olamiz.

Endi AB yoy og'irlik markazi sifatida $\max \Delta s_i \rightarrow 0$ dagi $(\bar{x}; \bar{y}) \rightarrow (x, y)$ nuqtani qabul qilsak (bunday chekli limit mavjud va u oraliqni bo'lish usuliga hamda i-oraliqdan olingan s'_i ning o'rniga bog'liq emas degan faraz asosida),

$$x = \frac{\int_0^S x(s)\rho(s)ds}{\int_0^S \rho(s)ds}, \quad y = \frac{\int_0^S y(s)\rho(s)ds}{\int_0^S \rho(s)ds}$$

ga ega bo'lamiz.

Maxrajdagi integral AB yoyning massasi ekanligini ko'rish qiyin emas, demak, oxirgilardan

$$x = \frac{1}{m} \int_0^S x(s)\rho(s)ds, \quad y = \frac{1}{m} \int_0^S y(s)\rho(s)ds, \quad m = \int_0^S \rho(s)ds \quad (12.7.30)$$

formulalarga kelimiz.

Agar AB yoy birjinsli (ya'ni uning zichligi ρ -o'zgarmas) bo'lsa, (12.7.30) (x, y) - og'irlik markazi uchun

$$x = \frac{1}{S} \int_0^S x(s)ds, \quad y = \frac{1}{S} \int_0^S y(s)ds \quad (12.7.31)$$

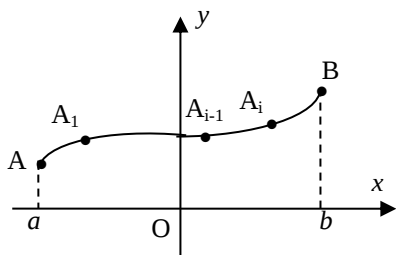
formulalarni olamiz

Mexanikada

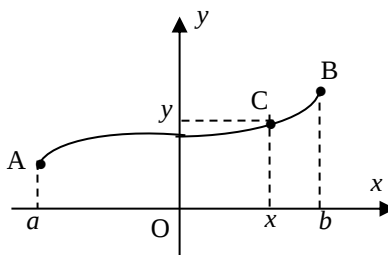
$$M_x = \int_0^S y(s)ds, \quad M_y = \int_0^S x(s)ds$$

integrallarni birjinsli ($\rho=1$ zichlik bilan) AB yoyning Ox va Oy o'qlarga nisbatan statik momentlari deb ham yuritiladi.

Aytaylik, birjinsli AB yoy $[a;b]$ kesmada uzluksiz differensiallanuvchi $y=f(x)$ funksiya grafigidan iborat bo'lsin (12.7.20- rasm).



12.7.19 rasm.



12.7.20 rasm.

U vaqtda

$$s = \int_a^x \sqrt{1 + (f'(\xi))^2} d\xi, \quad 0 \leq x \leq b,$$

$$ds = \sqrt{1 + (f'(x))^2} dx$$

bo'lib, (12.7.31) dan

$$S = \int_a^b \sqrt{1 + (f'(x))^2} dx,$$

$$x = \frac{1}{S} \int_a^b x \sqrt{1 + (f'(x))^2} dx, \quad (12.7.32)$$

$$y = \frac{1}{S} \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

formulalarni olamiz.

Agar birjinsli tekis AB yoy u yotgan tekislikdagi biror to'g'ri chiziqqa nisbatan simmetrik joylashgan bo'lsa, uning og'irlik markazi simmetriya o'qida yotishi tabiiydir. Haqiqatdan ham, AB yotgan tekislikda xOy Dekart koordinatalar sistemasini kiritib, Oy o'qni yoyning simmetriya o'qidan iborat qilib olsak va bu yoy $y=f(x)$ tenglamaga ega bo'lsa, u biror $[-a;a]$ kesmada juft funksiya dan iborat bo'ladi (12.7.21-rasm), agar $f'(x)$ ham $[-a;a]$ da uzluksiz deb faraz qilsak, $x\sqrt{1+(f'(x))^2}$ funksiya $[-a;a]$ da toq funksiya dan iborat bo'lib, yoy og'irlik markazining absissasi

$$X = \frac{1}{S} \int_{-a}^a x \sqrt{1 + (f'(x))^2} dx = 0$$

bo'ladi (kezi kelganda, simmetrik oraliq bo'yicha toq funksiya integrali nolga tengligini eslatamiz). Bu yoyning og'irlik markazi Oy o'qida, ya'ni yoyning simmetriya o'qida yotishini tasdiqlaydi. Bu holdan mexanikada amaliy masalalarni hal qilishda foydalaniladi.

1-misol. Radiusi r ga teng aylananing chorak qismidan iborat bo'lgan birjinsli yoyning og'irlik markazini toping (12.7.22- rasm).

Yechish. 12.7.22- rasmda AB aylana choragidan iborat yoy tasvirlangandir. Uning tenglamasi $y = \sqrt{r^2 - x^2}$, $0 \leq x \leq r$.

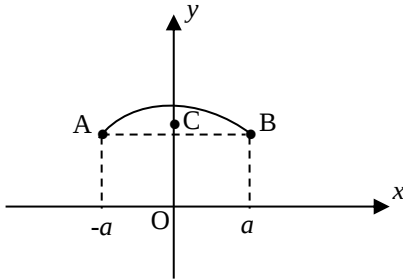
$$\text{Bundan } y' = -\frac{x}{\sqrt{r^2 - x^2}} = -\frac{x}{y}, \quad \sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{y^2}} = \frac{\sqrt{x^2 + y^2}}{y} = \frac{r}{y}.$$

Aylana chorak qismining uzunligi $S = \frac{1}{2}\pi r$ bo'lishi ma'lum. (12.7.32) ning oxirgi formulasidan.

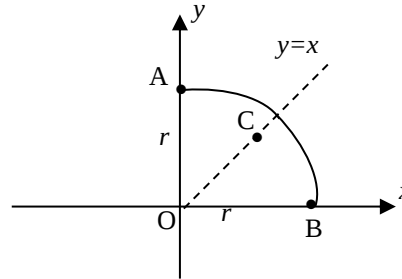
$$Y = \frac{1}{S} \int_0^r y \cdot \frac{r}{y} dx = \frac{2}{\pi r} \cdot r \cdot x \Big|_0^r = \frac{2r}{\pi}.$$

12.7.22-rasmdan ko‘rinadiki, AB yoy birinchi chorak bissektisasiga nisbatan simmetrikdir, demak, uning og‘irlik markazi shu bissektisada yotadi, ya’ni

$$X = Y = \frac{2r}{\pi}.$$



12.7.21-rasm.



12.7.22-rasm.

(12.7.32) ning oxirgi formulasidan

$$2\pi YS = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

tenglikni olish qiyin emas. Agar $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz differensiallanuvchi hamda manfiy emas deb faraz qilib, uning grafigi bo‘lgan AB yoyni Ox o‘qi atrofida aylantirishdan sirt hosil qilsak, bu tenglikning o‘ng tomoni shu aylanish sirtining yuzi ekanligi ma’lum ((12.7.23) formula), chap tomoni esa birjinsli AB yoy og‘irlik markazining Ox o‘qi atrofida aylanishidan hosil bo‘lgan aylana uzunligini yoy uzunligiga ko‘paytmasiga tengdir. Bu quyidagi teoremaning isbotidan iboratdir.

12.7.2-teorema (Guldining birinchi teoremasi). *Birjinsli tekis egri chiziqni u yotgan tekislikda olingan va uni kesmaydigan, biror o‘q atrofida aylantirish natijasida hosil qilingan aylanish sirtining yuzi shu egri chiziq yoyning uzunligi bilan uning og‘irlik markazi aylanishidan hosil bo‘lgan aylana uzunligining ko‘paytmasiga teng.*

2-misol. Radiusi r ga teng bo‘lgan birjinsli yarim aylananing og‘irlik markazi topilsin.

Yechish. Yarim aylanani 12.7.23-rasmdagidek joylashtirsak, Oy uning simmetriya o‘qidan iborat bo‘lib, og‘irlik markazining absissasi $X=0$ bo‘ladi. Ordinatasini topish uchun yuqoridagi teoremani qo‘llaymiz. Yarim aylana o‘z diametri atrofida aylanishidan sfera hosil bo‘lgani uchun va yuqoridagi teorema bo‘yicha

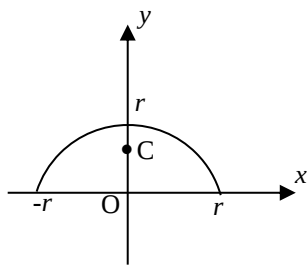
$$4\pi r^2 = 2\pi Y \cdot S, \quad S = \pi r^2 \Rightarrow 4\pi r^2 = 2\pi^2 r Y.$$

Bundan

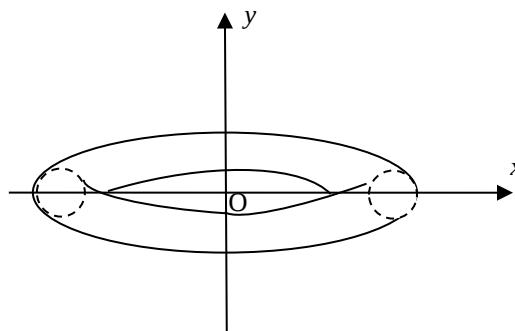
$$Y = \frac{2r}{\pi}.$$

Demak, 12.7.23-rasmdagi yarim aylana og‘irlik markazi $C\left(0; \frac{2r}{\pi}\right)$ nuqtadadir.

3-misol. Radiusi r ga teng bo‘lgan aylanani, uning tekisligida yotgan va uni kesmaydigan hamda aylana markazidan d masofada yotgan o‘q atrofida aylantirish natijasida hosil qilingan tor sirtining yuzi topilsin (12.7.24-rasm)



12.7.23-rasm.



12.7.24-rasm.

Yechish. Masala shartiga ko'ra $d \geq r$. Aylananing uzunligi $2\pi r$ va u birjinsli ekanligidan uning og'irlik markazi o'zining markazidan iborat bo'lib, og'irlik markazini Oy o'qi atrofida aylanishidan uzunligi $2\pi d$ ga teng bo'lgan aylana hosil bo'ladi. Demak, Guldinning birinchi teoremasiga ko'ra tor sirtining Q yuzi

$$Q = (2\pi r) \cdot (2\pi d) = 4\pi^2 r d$$

bo'ladi.

b) Tekis shaklning og'irlik markazi. Bu yerda $x=a$, $x=b$ ($a < b$) to'g'ri chiziqlar va $y=f_1(x)$, $y=f_2(x)$ egri chiziqlar bilan chegaralangan zichligi $\mu(x)$ bo'lgan tekis moddiy shaklni qaraymiz (12.7.25- rasmga qarang). Shu bilan birga $f_1(x) \geq f_2(x)$ va ular $[a;b]$ da uzluksiz deb faraz qilamiz. $[a;b]$ ni ixtiyoriy qilib n ta bo'laklarga bo'lamiz va bo'linish (tugun) nuqtalarini

$a = x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_{n-1} < x_n = b$ orqali belgilaymiz. Bu tugun nuqtalaridan Oy o'qqa parallel to'g'ri chiziqlar o'tkazib, tekis shaklni n ta bo'laklarga ajratamiz, i - bo'lakdan $\xi_i \in [x_{i-1}; x_i]$ nuqta tanlab bu bo'lakchani shakldagidek qilib, zichligi $\mu(\xi_i)$ gat eng bo'lgan birjinsli to'g'ri to'rt burchak bilan almashtiramiz. Agar ξ_i ni $[x_{i-1}; x_i]$ ning o'rtasi deb tanlasak, aytilgan to'g'ri to'rtburchakning og'irlik markazining koordinatalari

$$\xi_i = \frac{x_{i-1} + x_i}{2}, \quad h_i = \frac{f_1(\xi_i) + f_2(\xi_i)}{2};$$

uning yuzi esa $\Delta S_i = [f_2(\xi_i) - f_1(\xi_i)] \Delta x_i$ dan iborat bo'ladi ($\Delta x_i = x_i - x_{i-1}$). Agar bu to'rtburchakni o'zining og'irlik markazidagi moddiy nuqta bilan almashtirsak, olingan bunday moddiy nuqtalar sistemasining og'irlik markazi uchun

$$\bar{X} = \frac{\sum_{i=1}^n \mu(\xi_i) [f_2(\xi_i) - f_1(\xi_i)] \Delta x_i}{\sum_{i=1}^n \mu(\xi_i) [f_2(\xi_i) - f_1(\xi_i)] \Delta x_i},$$

$$\bar{Y} = \frac{\frac{1}{2} \sum_{i=1}^n \mu(\xi_i) [f_2^2(\xi_i) - f_1^2(\xi_i)] \Delta x_i}{\sum_{i=1}^n \mu(\xi_i) [f_2(\xi_i) - f_1(\xi_i)] \Delta x_i}$$

larni olamiz. Oxirgilarda $\max_i \Delta x_i \rightarrow 0$ limitga o'tib, tekis shakl og'irlik markazi (X, Y) uchun

$$X = \frac{\int_a^b \mu(x) [f_2(x) - f_1(x)] dx}{\int_a^b \mu(x) [f_2(x) - f_1(x)] dx}$$

$$Y = \frac{\frac{1}{2} \int_a^b \mu(x) [f_2^2(x) - f_1^2(x)] dx}{\int_a^b \mu(x) [f_2(x) - f_1(x)] dx}$$

larni olamiz va, nihoyat, birjinsli tekis shakl uchun

$$X = \frac{1}{S} \int_a^b x [f_2(x) - f_1(x)] dx, \quad Y = \frac{1}{2S} \int_a^b [f_2^2(x) - f_1^2(x)] dx \quad (12.7.33)$$

formulalar kelib chiqadi, bu yerda S tekis shaklning yuzi bo'lib,

$$S = \int_a^b [f_2(x) - f_1(x)] dx.$$

Bu o'rinda

$$M_x = \frac{1}{2} \int_a^b [f_2^2(x) - f_1^2(x)] dx$$

va

$$M_y = \int_a^b x [f_2(x) - f_1(x)] dx$$

larni berilgan birjinsli tekis shaklning mos ravishda Ox va Oy o'qlarga nisbatan statik momentlari deb yuritilishini ham aytamiz.

4-misol. Absissalar o'qi va $y=2x-x^2$ parabola bilan chegaralangan birjinsli shaklning og'irlik markazi topilsin (12.7.26- rasmga qarang).

Yechish.

$$2x - x^2 = 0 \Rightarrow a = 0, b = 2.$$

$$S = \int_0^2 (2x - x^2) dx = \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3};$$

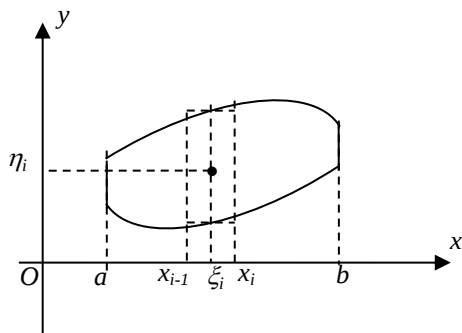
(12.7.33) ga ko'ra:

$$X = \frac{1}{S} \int_0^2 x(2x - x^2) dx = \frac{3}{4} \int_0^2 (2x^2 - x^3) dx = \frac{3}{4} \left(\frac{2}{3} x^3 - \frac{x^4}{4} \right) \Big|_0^2 = \frac{3}{4} \left(\frac{16}{3} - \frac{16}{4} \right) = \frac{3}{4} \cdot \frac{16}{12} = 1$$

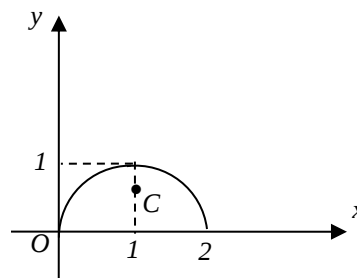
$$Y = \frac{1}{2S} \int_0^2 (2x - x^2)^2 dx = \frac{3}{2} \int_0^2 (4x^2 - 4x^3 + x^4) dx = \frac{3}{8} \left(\frac{4}{3} x^3 - x^4 + \frac{x^5}{5} \right) \Big|_0^2 = \frac{3}{8} \left(\frac{32}{3} - 16 + \frac{32}{5} \right) =$$

$$= \frac{3}{8} \left(10 \frac{2}{3} - 16 + 6 \frac{2}{5} \right) = \frac{3}{8} \cdot \left(\frac{2}{3} + \frac{2}{5} \right) = \frac{3}{4} \cdot \frac{8}{15} = \frac{2}{5} = 0,4.$$

Og'irlik markazi C(1;0,4) nuqtada. Bu misoldan ko'rinadiki, og'irlik markazi bu shaklning simmetriya o'qidadir.



12.7.25-rasm.



12.7.26-rasm.

Bu umumiy xossa bo'lib, agar birjinsli shakl biror o'qqa nisbatan simmetrik bo'lsa, uning og'irlik markazi shu simmetriya o'qida yotadi.

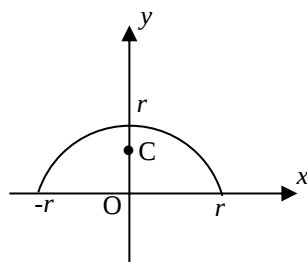
(12.7.33) formulalarning ikkinchisini

$$2\pi YS = \pi \int_a^b [f_2^2(x) - f_1^2(x)] dx$$

ko'rinishga keltirish qiyin emas. Bundan ko'rinadiki, tenglikning o'ng tomoni $\forall x \in [a; b]$ $0 \leq f_1(x) \leq f_2(x)$ faraz asosida tekis shaklni Ox o'qi atrofida aylantirishdan hosil bo'lgan jismning hajmidan iborat bo'lsa, chap tomoni esa birjinsli tekis shakl yuzini uning og'irlik markazi Ox o'qi atrofida aylanishidan hosil bo'lgan aylananing uzunligiga ko'paytmasidir. Ya'ni quyidagi o'rinli ekanligini oldik.

12.7.3-teorema (Guldenning ikkinchi teoremasi). Birjinsli tekis shaklni uning tekisligida yotgan va uni kesmaydigan o'q atrofida aylantirishdan hosil bo'lgan aylanish jismining hajmi uning yuzini og'irlik markazi aylanishidan hosil bo'lgan aylana uzunligiga ko'paytmasiga tengdir: $V=2\pi YS$.

5-misol. Radiusi r ga teng bo'lgan birjinsli yarim doiraning og'irlik markazini toping (12.7.27- rasmga qarang).



12.7.27-rasm.

Yechish. Agar yarim doirani 12.7.27- rasmdagidek joylashtirsak, u Oy o'qqa nisbatan simmetrikdir. Demak og'irlik markazi shu o'qda yotadi: $X=0$.

Guldenning ikkinchi teoremasi asosida

$$2\pi Y \cdot \frac{\pi r^2}{2} = \frac{4}{3} \pi r^3.$$

Bundan $Y = \frac{4}{3\pi} r$.

Demak, og'irlik markazi $C\left(0; \frac{4}{3\pi} r\right)$ nuqtada.

2. Inersiya momenti. Agar $A_1, A_2, \dots, A_n$ nuqtalarga joylashgan va massalari mos ravishda $m_1, m_2, \dots, m_n$ bo'lgan n ta moddiy nuqtalarning sistemasi berilgan bo'lib, nuqtalar $A_i(x_i; y_i)$ koordinatalarga ega bo'lsa, bu moddiy nuqtalar sistemasining Ox , Oy va koordinatalar boshiga nisbatan inersiya momentlari

I_x, I_y va I_0 lar (mos ravishda)

$$I_x = \sum_{i=1}^n m_i y_i^2, \quad I_y = \sum_{i=1}^n m_i x_i^2, \quad I_0 = \sum_{i=1}^n m_i (x_i^2 + y_i^2)$$

bo'ladi.

Oldingi banddagiga o'xshash ishlarni bajarib, (12.7.29) ko'rinishda berilgan birjinsli tekis egri chiziqli uchun

$$I_x = \int_0^s y^2(s) ds, \quad I_y = \int_0^s x^2(s) ds, \quad I_0 = \int_0^s (x^2(s) + y^2(s)) ds \quad (12.7.34)$$

flomulalarni olish mumkin.

6-misol. Birjinsli radiusi r ga teng bo'lgan yarim aylananing o'z diametriga nisbatan inersiya momenti topilsin.

Yechish. Yarim aylanani 12.7.23- rasmdagidek joylashtirib olsak, uning diametri Ox o'qiga joylashadi va Oy o'qqa nisbatan simmetrik bo'ladi. Uning tenglamasi

$$y = \sqrt{r^2 - x^2}, \quad -r \leq x \leq r.$$

$$y' = -\frac{x}{\sqrt{r^2 - x^2}} = -\frac{x}{y}, \quad ds = \sqrt{1 + y'^2} dx = \frac{r}{y} dx;$$

(12.7.34) ning birinchi formulasiga asosan:

$$I_x = \int_{-r}^r y^2 \cdot \frac{r}{y} dx = r \int_{-r}^r y dx = r \int_{-r}^r \sqrt{r^2 - x^2} dx = 2r \int_0^r \sqrt{r^2 - x^2} dx = 2r \cdot \frac{\pi r^2}{4} = \frac{\pi}{2} r^3.$$